Lecture notes on risk management, public policy, and the financial system

Interest rate and credit spread risk

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Interest rates

Interest rate risk

Credit spreads and spread risk

Interest rate risk measurement

Interest rates

Interest rates and yield curves

Bond math: spot, forward and par yield curves

Interest rate risk

Credit spreads and spread risk

Interest rate risk measurement

What are interest rates?

- An **interest rate** is a rate paid for borrowing money
  - Expressed as a percent of **principal**—amount of money borrowed—per unit of time, genrally per annum
  - E.g. 5 percent per annum→\$5 per annum for each \$100 of principal
- **Term** or **time to maturity** of the borrowing generally contractually specified
- Interest rate may be categorized by whether it varies over the term

  Fixed rate: set at a particular value for all or part of the term

  Floating rate: varies over the term by reference to an index

Fixed income instruments

Loans: bilateral contract between lender, generally a bank, and borrower

- Large loans may be **syndicated**, several lenders extend credit
- Secondary markets exist for syndicated and some other, generally large, loans , with credit extended by several lenders

Securities such as bonds, bills and notes

- Legal design facilitates offering them to many potential lenders at issuance and secondary market trading

Money markets: short-term lending, including

Bank deposits and other **unsecured interbank lending**

Repo and other **secured** forms ($\rightarrow$collateral markets)

Short-term securities such as Treasury bills, **commercial paper**

Derivatives including

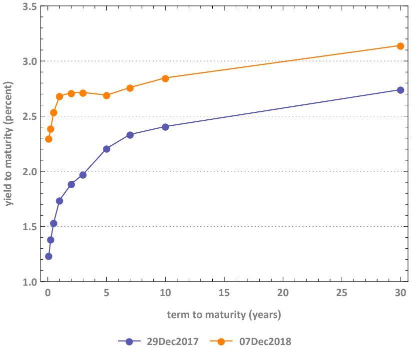
Money market and bond futures: exchange-traded contracts

Interest-rate swaps: regular exchange of fixed for floating payments, largest OTC fixed-income market

Credit default swaps: exchange of regular fixed payments for a par amount in the event of the default of an underlying security

Yield curve

- Interest rates—risk-free or for a given obligor—are not a single risk factor
- Typically vary by the term of the loan
- **Term structure of interest rates** or **yield curve:** rates as function of maturity



U.S. on-the-run Treasury yield curve. *Source*: Bloomberg Financial LP.

Typical behavior of the term structure

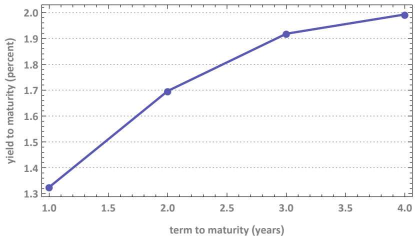
- Level and shape of curve based primarily on expected future short-term rates
- But also contains liquidity, credit, interest-rate and other risk premiums
- Term structure generally upward-sloping due to **risk premiums**
    - Longer-term risk-free interest rates and credit spreads generally higher than short-term
    - Particularly steep at very short end due to (→)**money premium**
- But may be downward-sloping—**yield curve inversion**—overall or in some segments, e.g.
    - Short-term rates spike, not expected to persist, e.g. emerging-market rates under foreign exchange pressure
    - High demand for safe long-term bonds, e.g. 2005 conundrum in U.S.

Equivalent ways of expressing the term structure

- Term structure can be represented in different ways, each useful in certain contexts
- **Yield curve:** yields of bonds/loans as a function of term
  - Yield-to-maturity calculations assume reinvestment of coupons at same yield
  - Ambiguity due to coupon size
- **Spot** or **zero-coupon curve** and **spot** or **zero-coupon rates:**
  - Money lent now and repaid at single specific time in the future
- **Forward curve** and **forward rates:**
  - Loans of a specified term to maturity commencing at different **settlement date** in the future
  - E.g. rates on 3-month loans settling immediately, in 1 month, in 3 months, ...
  - Can be transformed into spot curve and v.v.

Bond prices and yield to maturity

- Simple example:
  - Shortest-term issue: a 1-year bill or zero-coupon
  - Coupon bonds maturing in 2, 3 and 4 years
- Annualized interest rates, annual pay frequency and compounding



term	coupon	price	yield
1 year	0.00	98.692	1.325
2 years	1.75	100.104	1.697
3 years	2.00	100.237	1.918
4 years	2.00	100.029	1.992

Bond price, yield, and spot rate relationships

- Quote data generally in the form of prices and/or yields
- Price p_t of a t-year bond with coupon c_t is related to its (generally observable) yield y_t by

$$p_t = c_t \left[\frac{1}{1+y_t} + \frac{1}{(1+y_t)^2} + \cdots + \frac{1}{(1+y_t)^t} \right] + \frac{1}{(1+y_t)^t}$$

  - Converts quote in price terms to yield terms and v.v.
- Bond price also related to its (generally unobservable) spot rates $r_1, r_2, \ldots, r_t$:

$$p_t = c_t \left[\frac{1}{1+r_1} + \frac{1}{(1+r_2)^2} + \cdots + \frac{1}{(1+r_t)^t} \right] + \frac{1}{(1+r_t)^t}$$

  - Derives unobserved spot rates from observed quote data

Bootstrapping the spot curve from bond prices

- Derived yield curves used in pricing, valuation, risk analysis
  - Market quotes for different securities on same curve may employ different pay frequency, day count and other conventions
  - Derived curves—spot curve or other forms—impose consistency
- **Bootstrapping** a technique for obtaining spot curve, starting
  - Other techniques include spline interpolation, least-squares fitting
- Bootstrapping starts with shortest-maturity security, uses each successively longer security to capture one longer-term spot rate
- For each bond maturity, assumes we have either a price or yield (or both—and coupon rate)

Bootstrapping spot rates from prices: example

- 1-year bond has no coupon (i.e. $c_0 = 0$), so p_1 and r_1 satisfy

$$p_1 = 0.986923 = \frac{1}{1+r_1} = \frac{1}{1.01325}$$

- Next, solve for $r_2 = 0.0170$ from:

$$\begin{aligned} p_2 = 1.00104 &= c_2 \left[\frac{1}{1+r_1} + \frac{1}{(1+r_2)^2} \right] + \frac{1}{(1+r_2)^2} \\ &= \frac{0.0175}{1.01325} + \frac{0.0175+1}{(1+r_2)^2} \end{aligned}$$

- Solve the same type of equation to get $r_3 = 0.01925$:

$$\begin{aligned} p_3 = 1.00237 &= c_2 \left[\frac{1}{1+r_1} + \frac{1}{(1+r_2)^2} + \frac{1}{(1+r_3)^3} \right] + \frac{1}{(1+r_3)^3} \\ &= 0.02 \times 1.95377 + \frac{0.02+1}{(1+r_3)^3} \end{aligned}$$

- And, finally, $r_4 = 0.02$

Equivalent ways of expressing the spot curve

Discount curve and **discount factors:** present values corresponding to spot rates

- Discount factor corresponding to r_1 is $(1 + r_1)^{-1}$, discount factor corresponding to r_2 is $(1 + r_2)^{-2}$, etc.

Par yield curve: hypothetical bonds with coupons equal to yields

- Computed from spot curve by setting price to par and solving for coupon rate

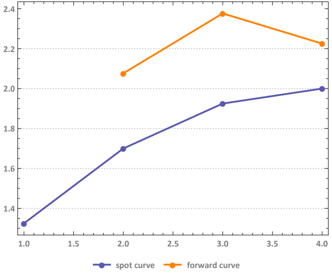
term	discount factor	par yield
1 year	0.9869	1.325
2 years	0.9668	1.697
3 years	0.9444	1.918
4 years	0.9238	1.993

Forward and spot rates: example

- Forward rate $f_{t,t+1}$ from t to $t+1$ related to r_t r_{t+1} by

$$f_{t,t+1} = \frac{(1+r_{t+1})^{t+1}}{(1+r_t)^t} - 1$$

- Forward rate $f_{0,1}$ settling immediately is identical to shortest-term spot rate
- Forward higher than spot rates if yield curve positively sloped
- Forward curve falling due to **convexity** of spot curve
  - Slope of spot curve positive but declining



Assumed spot and derived forward curves, percent.

term	spot	forward
1 year	1.325	
2 years	1.700	2.0764
3 years	1.925	2.3765
4 years	2.000	2.2253

Interest rates

Interest rate risk

- Sources of interest rate risk
- Inflation risk
- Annuities and market risk

Credit spreads and spread risk

Interest rate risk measurement

Sources of fixed income return

Deterministic sources: returns with an unchanged curve

Yield of the security

Rolldown or **theta:** term to maturity shortens over time as maturity date nears

- Generally positive, since yield curves generally upward-sloping
- **Example:** a 3-year bond held for 1 year becomes a 2-year bond, typically with a lower yield and higher market value

Interest rate risk: yield curve may change over time, affecting security values

- Changes in interest rates random and arise for many reasons

Decomposing interest rates

Risk-free or **pure rate of interest** compensates for passing of time

- **Expected future interest rates:** longer-term yield is based on short-term rates expected to prevail up to maturity
- **Term premium:** additional yield compensating for bearing interest-rate risk of holding longer-term securities
  - Aversion to term risk may change without change in expected future interest rates

Spreads over the risk-free rate compensate primarily for

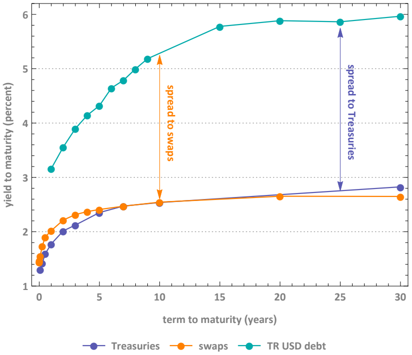
- **Credit risk:** compensation for risk of default and credit migration losses
- **Liquidity risk:** compensation for cost and risk of exiting, adjusting or maintaining position

Nominal rates express principal in units of money

- **Real rate** expressed in units of goods, e.g. gold or determined with reference to a price index
- **Inflation compensation** or **breakeven inflation** rate: difference between nominal and real rates

Spreads

- Risk-free curve: a base/benchmark relative to which risky yield curves measured
- Risky curve: generally expressed in terms of spreads relative to risk-free
- **Credit spread:** difference between interest rate on a risky and congruent credit risk-free security



Risk-free: U.S. on-the-run Treasury curve; risky: Republic of Turkey U.S. dollar-denominated yield curve, 16Jan2018. *Source*: Bloomberg Financial LP.

Attributing spreads to risks

- Each source of risk has an expected value and risk impact
- Wider spreads make bonds cheaper, and thus increase prospective future returns
  - Higher future return provides **risk premium** compensating for expected and unexpected losses
- Models needed to decompose spreads into various sources
- Spreads can also be measured across currencies

Credit risk premium

- Credit risk generally the main driver of spreads and changes in spreads over risk-free yields
- Credit spread compensates for expected value of default—default probability times loss given default
- But also compensates for "putting up with" default risk
- Includes the market risk of changes in credit-related portion of spread

Liquidity risk premium

- Liquidity risk can have countervailing impact on spreads
- **Liquidity premium:** aversion to risk of holding longer-term securities may change
  - More likely to affect corporate and sovereign issues outside advanced market economies
  - Sovereign issues of advanced market economies may have negative liquidity risk premium
- Financial stress likely to impair liquidity conditions, induce flight to quality, lowering yields on most-liquid sovereign bonds
- **Money premium** due to provision of money services by safe very short-term debt
  - Very short end of Treasury yield curve steeper than otherwise

Inflation risk

- **Inflation rate risk** is the risk of loss from a rise in the general price level
  - Directly affects securities with payoffs defined in nominal terms
  - Indirectly affects real assets by affecting macroeconomic conditions
- Inflation difficult to hedge
  - U.S. (since 1997), U.K., other countries issue **inflation-protected** or **inflation-linked bonds** that increase principal based on price index
  - **Inflation swaps** and other derivatives
  - **Gold clause** specifying gold value of principal and interest of certain U.S. Treasury bonds invalidated 1933 by Joint Resolution of Congress

Insurance company exposure to inflation

- Insurers may benefit from inflation
- Long-term liabilities generally defined in nominal terms
  - Generally not fully hedged against changes in interest rates
  - And substantial allocation to real assets: real estate, equities
- Permanent risk in inflation rate reduces real value of liabilities

Types of annuities

- **Annuities** are contracts for exchange of a specified sequence of payments between an **annuitant** and intermediary, generally an insurance company
- Very wide variety of types
- Payments by annuitant may be a lump sum or periodic over a future time interval
  - Annuities with periodic future payments may **lapse** or include **early surrender penalties**
- Payments by insurance company may be fixed or vary:

  Fixed annuity: payments or interest rate fixed over time

  Variable annuity: payments vary with return on a specified portfolio, generally equity-focused
- Annuities may include guarantees by insurance company, such as **guaranteed minimum benefits**

Risks of annuity issuance

- Market risks interact with risks arising from guarantees and policyholder behavior
- Variable annuities generally provide guaranteed minimum return
  - Economically equivalent to sale of put option on equity market by insurer to policyholder
  - Annuity is underpriced if value of put not fully incorporated
- Large losses to U.S. insurers in 2008
  - Hartford Life became a Troubled Asset Relief Program (TARP) recipient
- Fixed annuity issuance exposed to **convexity risk**
  - Assets generally duration-matched to liabilities
  - But liabilities exhibit greater convexity due to guarantees and policyholder behavior
  - Economically equivalent to sale of put option on bond market by insurer to policyholder
- Rising interest rates: early surrender optimal→duration falls rapidly
- Falling interest rates: minimum guaranteed rate in effect→duration rises rapidly

Interest rates

Interest rate risk

Credit spreads and spread risk
- Credit spreads
- Credit spread risk
- Credit spreads and hazard rates

Interest rate risk measurement

Credit spreads and credit quality

- Credit spreads vary across several dimensions: Differences in
  - Spreads of different obligors on bonds with similar credit quality
    - Third-party guarantees, collateral, position in ($\rightarrow$)capital structure
  - Spreads of same obligor on bonds with different credit quality
    - $\rightarrow$Credit spread of senior unsecured$<$subordinated debt
  - Spreads of same obligor on bonds with different maturities
- Credit spread compensates for several sources of risk
  - Credit risk, reflecting both default probability and expected loss given default
  - Market liquidity risk: many credit-risky bonds infrequently traded or have small issuance volume

Measuring credit spreads

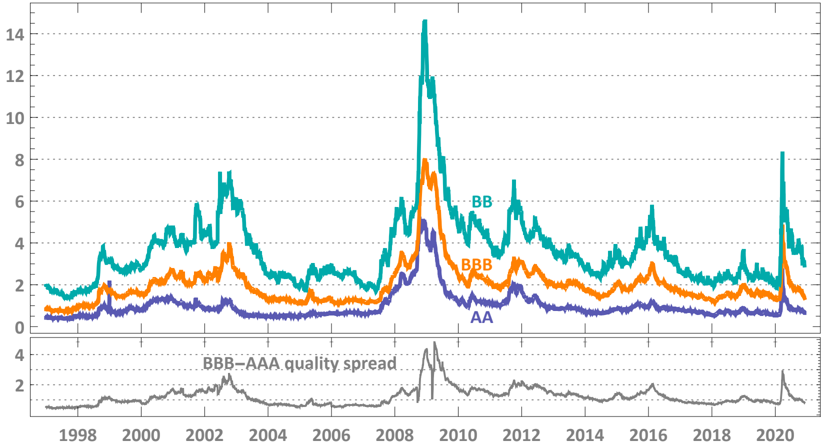
Spread to swaps or Treasuries with similar maturity. Can be implemented in different ways, e.g.

- *z*-**spread**: spread between zero-coupon curves
- *i*-**spread**: spread over interpolated risk-free curve

Option-adjusted spread (OAS): spread to a benchmark curve after accounting for the value of options embedded in the security

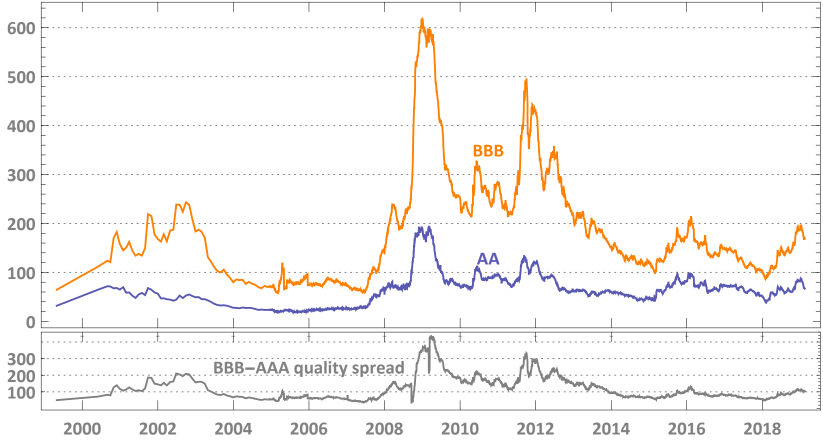
- Corporate bonds often **callable**
- Callable bond is bundled with call option on bond sold by investor to issuer

U.S. credit spreads 1996–2019



ICE BofA Merrill Lynch U.S. indexes of option-adjusted spreads (OAS) to the Treasury curve for corporate securities rated AA (C0A2), BBB (C0A4) and BB (H0A1), in percent. The BBB-AAA quality spread is the difference between the BBB and AA (C0A1) indexes. Daily, 31Dec1996 to 14Feb2019, downloaded from FRED.

European credit spreads 1999—2019



Bloomberg Barclays Euro Corporate Bond Index option-adjusted spread (OAS) to the treasury curve for securities rated AA and BBB, in basis points. The BBB-AAA quality spread is the difference, in percent, between the BBB and AAA indexes. Daily, 30Apr1999 to 15Feb2019, downloaded from Barclays Live.

Credit derivatives

Credit default swaps (CDS): OTC contract, one counterparty pays regular premium and is made whole by the other if default of specific bond issuer occurs

- Economically similar to guarantee or insurance on firm's debt, but without requirement of direct economic exposure
- Partial standardization: **single-name CDS** governed by master agreements
- Pre-crisis, large volumes of CDS on residential and commercial real-estate securitizations

Credit default swap indexes (CDX)

- Protection on a set of **reference entities**
- **Examples**: CDX.IG.NA series covering North America, iTraxx series covering Europe, Asia

Credit default swap derivatives based on underlying CDS or CDX

- Standard tranches ($\rightarrow$structured credit products)
- OTC swaptions on CDS and CDX

Pricing of credit derivatives

CDS spread: market-clearing premium in basis points paid for default protection

- Investment-grade CDS quoted in terms of credit spread
- High-yield CDS quoted in **points up front:** spread is standardized to 100 or 500 basis points, an upfront fee equates present value to market-based credit spread

CDS basis: difference between the CDS and bond spread

U.S. credit default swap index 2011–2019

On-the-run 5-year Markit CDX North American Investment Grade index, in basis points. Daily, 09Sep2011 to 02Jan2019. *Source*: Bloomberg Financial LP.

Credit spread risk

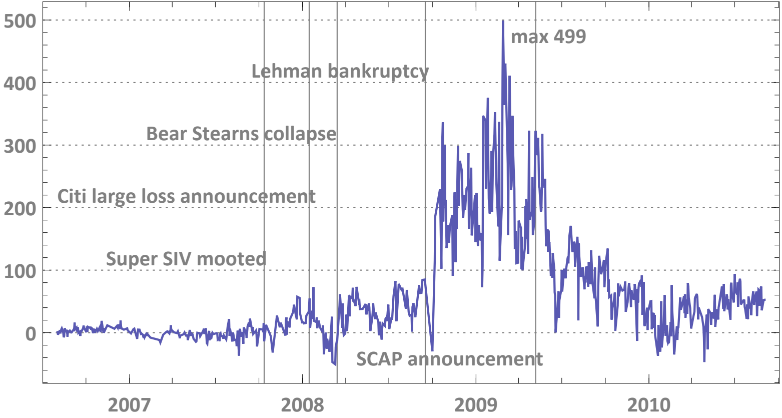
- **Credit spread risk**: risk of loss from change in credit spreads
  - Spread risk is a *market* risk, albeit credit-related
  - **Spread 01**: price/value impact of change in spreads, market risk of credit exposures
- Both the spread and spread risk a function of obligor credit quality
  - As well as of bond tenor, liquidity and other factors

Some special types of credit spread risk

Basis risk Risk that two similar, but not identical, securities diverge or converge in price.
Example: **Bond-CDS basis**, difference between credit spreads implied by bonds and CDS, influenced by availability of funding

Convexity risk Risk of loss from under-hedging a security with a highly non-linear payoff profile.
Example: **Structured credit products**, equity tranche may have highly non-linear response to change in default correlation (May 2005)

Citigroup CDS basis 2007–2010



Difference between the *z*-spread over Libor of Citigroup Inc. bonds and the premium on Citigroup 10-year senior unsecured CDS. The bond spread is blended from spreads on two senior unsecured issues: the 4.7% maturing May 29, 2015 (CUSIP 172967CY5) and the 5.85% maturing Aug. 2, 2016 (CUSIP 172967DQ1). *Source*: Bloomberg Financial LP.

Estimating default probability from credit spreads

- Credit spread expresses **risk-neutral hazard rate** and **default probability**
- Simplest case:
  - 1-year risk-free with yield r and credit-risky bonds with yield $r+z$ both exist
  - Recovery rate zero
- These present values should be equal:

$$\frac{(1-\tilde{\pi}_1)\times 1+\tilde{\pi}_1\times 0}{1+r}=\frac{1}{1+r+z}$$

- Spread z approximately equal to 1-year default probability $\tilde{\pi}_1$:

$$\boxed{\tilde{\pi}_1 \approx z}$$

Default probability and the hazard rate

- Credit risk models typically in continuous-time setting
- With r the risk-free continuously-compounded rate
    - z is the spread and $r + z$ is the continuously-compounded risky yield
    - Risk-neutral 1-year hazard rate $\tilde{\lambda}$:

$$\tilde{\lambda} = z$$

    - Via $e^a - 1 \approx a$
    - Approximation reasonably accurate for coupon securities, longer maturities

Credit spreads and recovery rates

- With recovery rate $R > 0$, credit spread approximately equal to expected loss (EL)

$$z \approx \tilde{\pi}_t(1 - R)$$

- If $R > 0$, risk-neutral default probability *higher*:

$$\boxed{\tilde{\pi}_1 \approx \frac{z}{1 - R}}$$

  - If you observe a given spread, then the lower the LGD, the higher must be the default probability

- **Example** (all in percent):

Credit spread	z	2.00
1-yr. risk-neutral default probability ($R = 0$)	$\tilde{\pi}_1 \approx z$	2.00
1-yr. risk-neutral default probability ($R = 0.40$)	$\tilde{\pi}_1 \approx \frac{z}{1-R}$	3.33

Interest rates

Interest rate risk

Credit spreads and spread risk

Interest rate risk measurement
- Measuring bond price sensitivity to rates
- Duration and convexity
- Measuring Value-at-Risk for a bond position

Bond and derivatives exposure to interest rates

- Interest rate risk can be measured as exposure to
  - **Parallel shifts** of entire yield curve
  - **Yield to maturity:** closed-form measures in some cases
  - **Key rates:** impact of changes in rates for specific maturities, e.g. 3-month or 2-year rates
    - Useful for hedging, bond portfolio management
- Complexities introduced by need to measure impact of changes in *rates* on *price* or *value*

Scenario analysis

- Analysis in which yield curve assumed to change in specific ways
- Can be carried out using expression for price p_t of a bond in terms of spot rates
  - Scenario result is price change $\tilde{p}_t - p_t$ resulting from specified changes in spot rates
- Some commonly-encountered scenario analyses include

Parallel shift: measure price change if all spot rates rise, say, 25 basis points

Curve steepening: longer-term risk-free rates rise, but short-term rates and credit spreads unchanged

Credit spread widening: credit spreads increase, but risk-free rates unchanged

- Change in *price* is the same for equal change in risk-free rate or spread

Roll-down return: bond "ages" by, say, 1 year, but rates and spreads unchanged

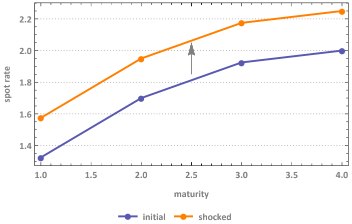
Scenario analysis: parallel shift

- Price change $\tilde{p}_t - p_t$ if all spot rates rise 25 basis points:

$$\tilde{p}_t = c_t \left[\frac{1}{1 + r_1 + 0.0025} + \cdots + \frac{1}{(1 + r_t + 0.0025)^t} \right] + \frac{1}{(1 + r_t + 0.0025)^t}$$

- All bond prices fall, but longer-term bond prices fall most
- In the **example:**

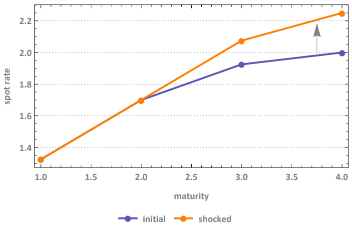
term	initial p_t	shocked $\tilde{p}_t$	$\tilde{p}_t - p_t$
1 year	98.692	98.449	-0.246
2 years	100.104	99.618	-0.486
3 years	100.237	99.517	-0.718
4 years	100.029	99.082	-0.946



Scenario analysis: curve steepening

- **Example:** 1- and 2-year spot rates unchanged, 3- and 4-year spot rates rise by 15 and 25 bps (0.0015 and 0.0025 percent)
- Short-term bond prices unchanged, but longer-term bond prices fall

term	initial p_t	shocked $\tilde{p}_t$	$\tilde{p}_t - p_t$
1 year	98.692	98.692	0.000
2 years	100.104	100.104	0.000
3 years	100.237	99.813	-0.423
4 years	100.029	99.102	-0.926



Scenario analysis: roll-down

- Assume spot curve unchanged 1 year hence
- Short-term bonds will return exactly their initial yield to maturity
- If yield curve upward-sloping, all bonds will experience positive return
  - As maturities shorten, bonds' cash flows discounted at lower spot rates
- In the **example:**

term	initial	1 year hence	Δ price
Initially 1 year	98.692	100.000	1.325
Initially 2 years	100.104	100.419	0.315
Initially 3 years	100.237	100.592	0.355
Initially 4 years	100.029	100.237	0.208

Rate sensitivity using duration and convexity

- Approximate measures of impact of small changes in yield y_t (or of small parallel shifts) on bond price p_t:

  Modified duration: denoted $\text{mdur}_t \equiv -\frac{1}{p_t}\frac{dp_t}{dy_t}$

  - Percent change in bond price as yield or level of curve rises 1 percent
  - Negative relation between price and yield$\rightarrow$convention of multiplying by -1

  Convexity: denoted $\text{conv}_t \equiv \frac{1}{p_t}\frac{d^2p_t}{dy_t^2}$

  - Change in duration as yield or level of curve changes

- Effect on price/value of a Δy increase in yield in bps is Taylor-approximated by

$$\boxed{\frac{\Delta p_t}{p_t} \approx -\,\text{mdur}_t\,\Delta y + \frac{1}{2}\,\text{conv}_t\,\Delta y^2,}$$

  with mdur_t the delta, conv_t the gamma of a bond

Calculating duration

- **Effective duration:** approximate $\frac{dp_t}{dy_t}$ by shifting curve up and down by 1 basis point:

$$\frac{dp_t}{dy_t} \approx \frac{p_t(\text{shifted up}) - p_t(\text{shifted down})}{2 \times 0.0001}$$

  - Divide by initial bond price (and express as a positive number)

term	initial	$+1$bp shock	-1bp shock	duration
1 year	98.692	98.683	98.702	0.987
2 years	100.104	100.084	100.123	1.950
3 years	100.237	100.208	100.266	2.886
4 years	100.029	99.991	100.067	3.807

- Why use effective duration?
  - Often easier to implement
  - Needed for securities with cash flows that depend on interest rates, e.g. mortgage-backed securities (prepayments), bonds with embedded options
- **Effective convexity** estimated analogously

Duration and convexity of U.S. 10-year note

Screen capture of Bloomberg Quick Yield Analysis for U.S. on-the-run 10-year note on 17Jan2017.

Example: rate sensitivity of U.S. 10-year note

- 2 percent coupon, expiration 15Nov26 (term 9 year 10 months)
- Price quoted in dollars, 32nds and 64ths of a dollar per \$100 of par value:

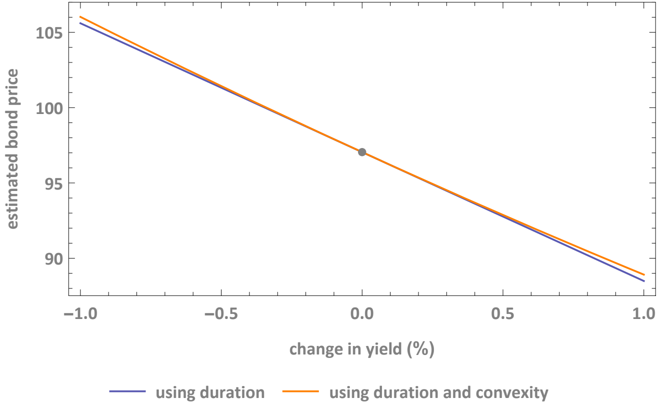
$$97\text{-}01+ \equiv 97 + \frac{1}{32} + \frac{1}{64} = 97.0468750$$

  - This quote includes 01 32nds; the + indicates an additional 64th
- Bloomberg estimate of modified duration $\text{mdur}_t = 8.8160$
  - If yield falls by 1 bp, price rises by 0.088160 percent

$$97.0468750 \times (1 + 0.088160) = 97.1324$$

- Bloomberg estimate of convexity $\text{conv}_t = 0.8713$
  - **positive convexity:** duration declines slightly as yield rises, attenuates price decline
  - Note: Bloomberg divides conventional convexity measure by 100
  - Makes it easier to work with percent rather than decimal changes
- Some types of bonds, e.g. mortgage-backed securities, display large-magnitude **negative convexity**
  - Behaves like negative option gamma, increases risk

Example: response of U.S. 10-year note to shocks



- Impact of 25 bps increase in rates
  - Duration: $-\,\mathrm{mdur}_t\, \Delta y = -8.816 \cdot 0.0025 = -0.0220$ (-2.20 percent)
  - Convexity: $\frac{1}{2}\, \mathrm{conv}_t\, \Delta y^2 = 87.13 \cdot 0.0025^2 = 0.00027$ (0.027 percent)
  - Total: $-8.816 \cdot 0.0025 + 87.13 \cdot 0.0025^2 = -0.0217$ (-2.17 percent)

Interest rate volatility

- Value-at-Risk (VaR) measure requires interest rate volatility estimate
- Two standard conventions in fixed-income option markets for implied volatility

  Normal volatility: standard deviation of changes in yield $\sigma_{\mathrm{n},t}$ in basis points

  Yield or **Black volatility:** standard deviation of proportional changes in yield $\sigma_{\mathrm{y},t}$ in percent

  - Divide normal vol by yield:

$$\sigma_{\mathrm{y},t} = \frac{\sigma_{\mathrm{n},t}}{y_t}$$

  - Measures yield changes—oddly—in percent rather than bps
  - But avoids—perhaps unnecessarily—negative yield scenarios
- Implied volatilities are ($\rightarrow$)risk-neutral estimates
  - Reflect market information, but embed risk premiums embedded in option prices

Applying interest rate volatility

- **Example:** typical (as of late 2016) at-the-money short- to medium-term swaption on 10-year USD swap
  - Normal volatility of about 90bps at annual rate
  - Swap rate of 2.25 percent
  - $\Rightarrow$Yield volatility of about $\frac{90}{2.25} = 40$ percent at annual rate
  - Or $\frac{0.4}{\sqrt{256}} = 0.025$ (2.5 bps) per trading day applying ($\rightarrow$)square-root-of-time rule

Quantiles of the distribution of yield

- Assume yield lognormally distributed
  - $\Rightarrow$Proportional yield changes normally distributed
- Compute return quantiles using yield volatility estimate, appropriate normal quantile, and time horizon τ:
  - z_α : α-quantile of standard normal distribution
  - $\sigma_{\mathrm{y},t}$: statistical or risk-neutral estimate of itemize volatility
- Estimated α-quantile of proportional change in yield from t to $t+\tau$ is then $e^{z_\alpha \sigma_t \sqrt{\tau}} - 1$
  - Arithmetic approximation: $z_\alpha \sigma_{\mathrm{y},t}$
- Multiply VaR quantile by yield to obtain yield changes in basis points $y_t \left(e^{z_\alpha \sigma_t \sqrt{\tau}} - 1\right)$
  - Equivalent to using normal vol, hence the name

$$z_\alpha \sigma_{\mathrm{y},t} \sqrt{\tau} y_t = z_\alpha \sigma_{\mathrm{n},t} \sqrt{\tau}$$

Computing VaR for a bond

- If the model is accurate, the probability is α that yield will rise (fall) no more than $100z_\alpha\sigma_{\mathrm{y},t}$ ($100z_{1-\alpha}\sigma_{\mathrm{y},t}$) percent
- Use upper (lower) tail of normal distribution for long (short) bond position
    - Since bond value declines as rates rise
- Apply ($\rightarrow$)delta-gamma approach, with duration and convexity in roles of delta and gamma
    - Set Δy to the VaR quantile of the change in yield in bps
- VaR—in percent of value—of a long position is then (using normal vol)

$$\mathrm{mdur}_t\, z_\alpha\sigma_{\mathrm{n},t}\sqrt{\tau} - \frac{1}{2}\,\mathrm{conv}_t(z_\alpha\sigma_{\mathrm{n},t}\sqrt{\tau})^2$$

    - The signs are reversed to convert the negative impact of a rise in yield to a positive number

Example: VaR of long U.S. 10-year note position

- Assume yield (0.023381 in example) lognormally distributed, proportional yield changes then normal
- Use ballpark estimate of normal vol drawn from option markets—90 bps per annum—to compute yield volatility and VaR shock
    - $\Rightarrow$Yield volatility $\frac{0.0090}{0.023381} = 0.384928$ or 38.5 percent at annual rate
- One-day return quantile of order 0.01 (99 percent confidence level):
    - Divide annual vol by $\sqrt{256} = 16$, multiply by $z_\alpha = 2.32635$
    - $\Rightarrow$Yield volatility $2.32635\frac{0.384928}{16} = 0.0559673$ or a 5.6 percent change in yield
    - VaR quantile in basis points is $2.32635\frac{0.384928}{16}0.023381 = 2.32635\frac{0.0090}{16} = 0.00130857$
- VaR in dollars: initial bond value times proportional change in value using pertinent VaR yield change quantile

$$\left(8.816 \cdot 0.00131 - \frac{1}{2}87.13 \cdot 0.00131^2\right) \times 970\,468.75 = 11\,123.30$$